Surface-links which bound immersed handlebodies (Intelligence of Low-dimensional Topology)

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数理解析研究所講究録
第2004巻 2016年 30-37

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1 Introduction

This article is a summary for ribbon-clasp surface-links defined in our paper [4], which are generalization of ribbon surface-links.

Throughout this article, we work in the PL or smooth category. An *immersed surface-link* or simply a *surface-link* means a closed oriented surface immersed in $\mathbb{R}^4$ such that each multiple point is a transverse double point. In the PL category, we assume that immersions are locally flat. When it is embedded, we also call it an *embedded surface-link*. Two surface-links are said to be *equivalent* if they are ambient isotopic.

A surface-link is *trivial* if it bounds a disjoint union of handlebodies embedded in $\mathbb{R}^4$. In particular, a *trivial 2-link* means the boundary of a disjoint union of 3-balls embedded in $\mathbb{R}^4$. A surface-link is *ribbon* if it bounds a disjoint union of handlebodies immersed in $\mathbb{R}^4$ whose multiple point set consists of ribbon singularities. (Note that a ribbon surface-link is an embedded surface-link.) A surface-link is *ribbon-clasp* if it bounds a disjoint union of handlebodies immersed in $\mathbb{R}^4$ whose multiple point set consists of ribbon singularities and clasp singularities. We give definitions of a ribbon singularity and a clasp singularity in Section 2. (For an immersion $f : M \to \mathbb{R}^4$ of a compact 3-manifold M, the *boundary* of the immersed 3-manifold $f(M)$ means the image $f(\partial M)$ of the boundary ∂M of M.)

In this article, we show two characterizations of a ribbon-clasp surface-link. First, we characterize it in terms of 1-handle surgeries and finger moves (Theorem 3.2). Second, we characterize it in terms of normal forms for immersed surface-links (Theorem 4.4). We introduce 1-handle surgeries and finger moves in Section 3 and normal forms for immersed surface-links in Section 4.

2 Ribbon singularities and clasp singularities

In this section, we explain a ribbon singularity and a clasp singularity.

Let M be a compact 3-manifold with non-empty boundary and $f : M \to \mathbb{R}^4$ an immersion of M into $\mathbb{R}^4$. Let Δ be a connected component of the multiple point set $\{x \in f(M) \mid \# f^{-1}(x) \geq 2\} \subset \mathbb{R}^4$.

We say that Δ is a *ribbon singularity* if Δ is a 2-disk in $\mathbb{R}^4$ and the preimage of Δ is the disjoint union of embedded 2-disks Δ_1 and Δ_2 in M such that Δ_1 is properly embedded in M and Δ_2 is embedded in the interior of M. Figure 1 shows a local model of a ribbon singularity in the motion picture method.

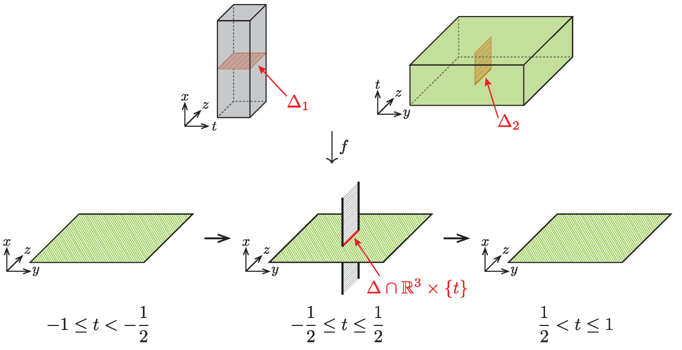

Figure 1: A local model of a ribbon singularity

We say that Δ is a *clasp singularity* if Δ is a 2-disk in $\mathbb{R}^4$ and the preimage of Δ is the disjoint union of embedded 2-disks Δ_1 and Δ_2 in M such that for each $i \in \{1, 2\}$, $\partial\Delta_i$ is the union of two arcs α_i and β_i, where α_i is a properly embedded arc in M and β_i is a simple arc in ∂M which connects endpoints of α_i. Figures 2 and 3 show local models of a clasp singularity in the motion picture method.

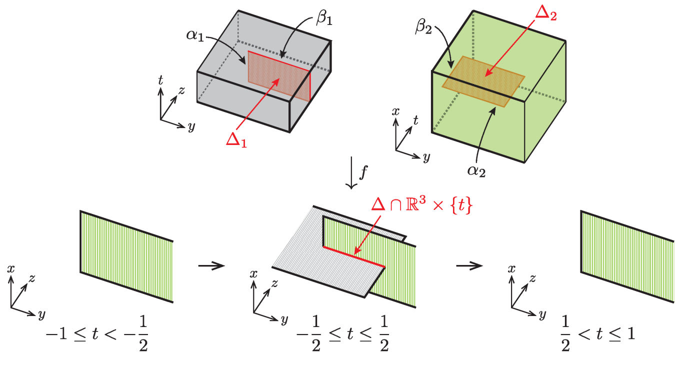

Figure 2: A local model of a clasp singularity

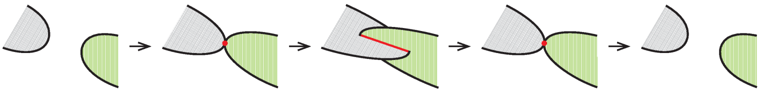

Figure 3: Another local model of a clasp singularity

Example 2.1 A Montesinos twin is a surface-link which is the boundary of a pair of embedded 3-disks B_1 and B_2 with a single clasp singularity between B_1 and B_2. Figure 4 shows a Montesinos twin $T = S_1 \cup S_2$, where $S_i = \partial B_i$ $(i \in \{1, 2\})$.

A Montesinos twin has two double points with opposite signs. Note that the equivalence class, as a surface-link, of a Montesinos twin is unique.

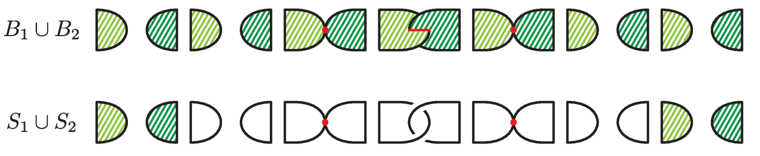

Figure 4: A Montesinos twin $T = S_1 \cup S_2$

Definition 2.2 *An M-trivial* 2*-link is a split union of a trivial* 2*-link and some (or no) Montesinos twins.*

3 1-handle surgeries and finger moves

Let F be a surface-link. A *chord* attached to F means an unoriented simple arc γ in $\mathbb{R}^4$ such that $F \cap \gamma = \partial\gamma$, which misses the double points of F. Two chords attached to F are *equivalent* if they are ambient isotopic by an isotopy of $\mathbb{R}^4$ keeping F setwise fixed.

A 1*-handle* attached to F means an embedded 3-disk B in $\mathbb{R}^4$ such that $F \cap B$ is the union of a pair of mutually disjoint 2-disks in ∂B, $F \cap B$ misses the double points of F, and the orientation of $F \cap B$ induced from ∂B is opposite to the orientation induced from F. Put

$$\mathrm{h}(F; B) := \mathrm{Cl}(F \cup \partial B - F \cap B),$$

which we call the surface-link obtained from F by a 1*-handle surgery* along B. (Here, Cl means the closure.) Two 1-handles attached to F are said to be *equivalent* if they are ambient isotopic by an isotopy of $\mathbb{R}^4$ keeping F setwise fixed. It is known [1, 3] that 1-handles attached to F are equivalent if and only if their cores are equivalent as chords attached to F. For a chord γ attaching to F, we denote by $\mathrm{h}(F;\gamma)$ the surface-link obtained from F by a 1-handle surgery along a 1-handle whose core is γ. Figure 5 shows a local model of a 1-handle surgery along γ.

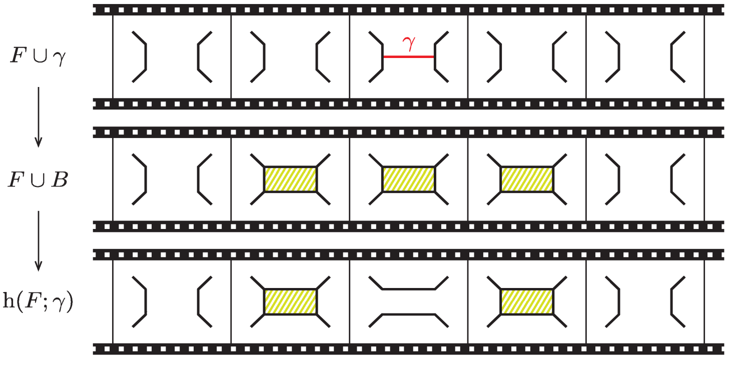

Figure 5: A local model of a 1-handle surgery along γ

A finger move is the inverse operation of the Whitney trick; for details, see [2, 7]. We give an alternative definition of a finger move by using a Montesinos twin and 1-handle surgeries as follows.

Let F be a surface-link and U a 4-disk in $\mathbb{R}^4$ disjoint from F. Put a Montesinos twin $T = S_1 \cup S_2$ in U and let B_1 and B_2 be embedded 3-disks in U with a single clasp singularity with $S_i = \partial B_i$ $(i \in \{1, 2\})$. Take a point $p_1 \in S_1 - S_1 \cap S_2$, a point $p_2 \in S_2 - S_1 \cap S_2$ and two distinct points q_1, q_2 in F missing the double points of F. Let γ_1 and γ_2 be oriented chords attached to $F \cup T$ such that for $i \in \{1, 2\}$, γ_i starts from p_i and terminates at q_i and $\gamma_i \cap (B_1 \cup B_2) = \{p_i\}$. Let F' be the surface-link obtained from $F \cup T$ by 1-handle surgeries along two 1-handles whose cores are γ_1 and γ_2. Let γ be a chord attached to F which is the concatenation of γ_1^{-1}, a simple arc from p_1 to p_2 in U and γ_2. We say that F' is obtained from F by a *finger move* along γ, which we denote by $\mathrm{f}(F; \gamma)$. It is seen that if γ and γ' are equivalent chords attached to F then $\mathrm{f}(F; \gamma)$ is equivalent to $\mathrm{f}(F; \gamma')$. Figure 6 shows a local model of a finger move along γ, where two 1-handles attached to $F \cup T$ whose cores are γ_1 and γ_2 are omitted for simplicity.

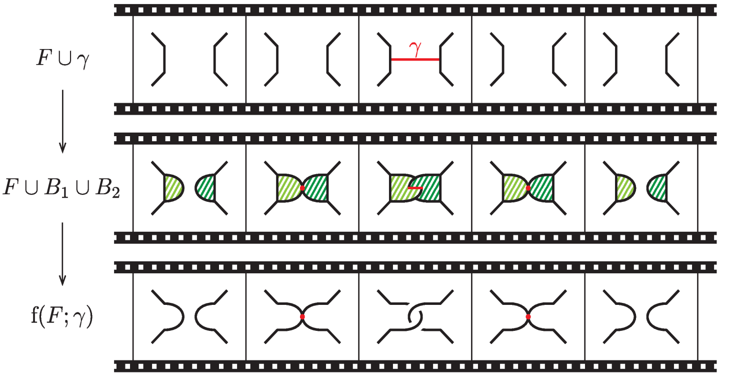

Figure 6: A local model of a finger move along γ

In terms of 1-handle surgeries and finger moves, ribbon surface-links and ribbon-clasp surface-links are characterized as follows.

Theorem 3.1 ([6, 8]) *A surface-link is ribbon if and only if it is obtained from a trivial 2-link by 1-handle surgeries.*

Theorem 3.2 ([4]) *For a surface-link F, the following conditions are equivalent.*

(1) F is a ribbon-clasp surface-link.

(2) F is obtained from a ribbon surface-link by finger moves.

(3) F is obtained from a trivial 2-link by 1-handle surgeries and finger moves.

(4) F is obtained from an M-trivial 2-link by 1-handle surgeries.

It is seen that Theorem 3.2 is a generalization of Theorem 3.1.

4 Normal forms for surface-links

4.1 Normal forms for embedded surface-links

Let L be an oriented link in $\mathbb{R}^3$. A *band* attached to L means an oriented 2-disk B in $\mathbb{R}^3$ such that $L \cap B$ is the union of a pair of mutually disjoint arcs in ∂B and the orientations of $L \cap \partial B$ induced from ∂B and L are opposite. We say that a link L' is obtained from L by a *band surgery* along B if $L' = \mathrm{Cl}(L \cup \partial B - L \cap \partial B)$, see Figure 7. Let $\mathcal{B} = B_1 \cup \ldots \cup B_m$

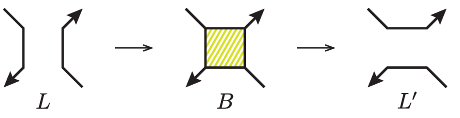

Figure 7: A band surgery from L to L' along a band B

be mutually disjoint oriented 2-disks in $\mathbb{R}^3$ such that each B_i is a band attached to L. A band surgery from L to L' along $\mathcal{B}$ is denoted by $L \xrightarrow{\mathcal{B}} L'$ or simply $L \to L'$.

For a band surgery $L \xrightarrow{\mathcal{B}} L'$, the *realizing surface* is a compact oriented surface, say F, properly embedded in $\mathbb{R}^3 \times [a, b]$ defined by:

$$F \cap \mathbb{R}^3 \times \{t\} = \begin{cases} L' \times \{t\} & \text{for } t \in ((a+b)/2, b] \\ L \cup \mathcal{B} \times \{t\} & \text{for } t = (a+b)/2 \\ L \times \{t\} & \text{for } t \in [a, (a+b)/2). \end{cases}$$

This realizing surface is denoted by $F(L \xrightarrow{\mathcal{B}} L')_{[a,b]}$.

Let $\mathcal{L} : L_1 \to L_2 \to \ldots \to L_m$ be a band surgery sequence. The *realizing surface* $F(\mathcal{L})_{[a,b]}$ of $\mathcal{L}$ in $\mathbb{R}^3 \times [a, b]$ with a division $a = t_1 < t_2 < \ldots < t_m = b$ is the union of the realizing surfaces $F(L_i \to L_{i+1})_{[t_i,t_{i+1}]}$ for $i = 1, ..., m-1$, see the left of Figure 8. (Note that the ambient isotopy class of the realizing surface $F(\mathcal{L})_{[a,b]}$ does not depend on the choice of a division.) If the links L_1 and L_m are trivial links, then there exist disk systems $\mathcal{D}$ and $\mathcal{D}'$ in $\mathbb{R}^3$ with $\partial\mathcal{D} = L_1$ and $\partial\mathcal{D}' = L_m$. Then we obtain a closed oriented surface

$$\overline{F}(\mathcal{L})_{[a,b]} := \mathcal{D} \times \{a\} \cup F(\mathcal{L})_{[a,b]} \cup \mathcal{D}' \times \{b\}$$

in $\mathbb{R}^3 \times [a, b]$, which we call the *closed realizing surface* of $\mathcal{L}$, see the right of Figure 8. Note that by Horibe-Yanagawa's lemma shown in [5], the equivalence class of $\overline{F}(\mathcal{L})_{[a,b]}$ does not depend on choices of disk systems $\mathcal{D}$ and $\mathcal{D}'$. We say that an embedded surface-link is in a *normal form* if it is a closed realizing surface $\overline{F}(\mathcal{L})_{[a,b]}$ of a band surgery sequence $\mathcal{L}$.

Theorem 4.1 ([5]) *Every embedded surface-link with μ components and g total genus is equivalent to the closed realizing surface of a band surgery sequence*

$$O \to L_- \to L_0 \to L_+ \to O',$$

where O and O' are trivial links, L_- and L_+ are μ-component links and L_0 is a $(\mu+g)$-component link.

A ribbon surface-link is characterized in terms of normal forms as follows.

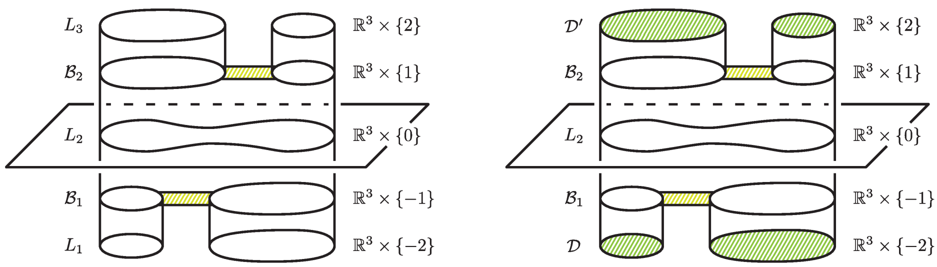

Figure 8: The realizing surface $F(L_1 \xrightarrow{\mathcal{B}_1} L_2 \xrightarrow{\mathcal{B}_3} L_3)_{[-2,2]}$, and the closed realizing surface $\overline{F}(L_1 \xrightarrow{\mathcal{B}_1} L_2 \xrightarrow{\mathcal{B}_2} L_3)_{[-2,2]}$

Theorem 4.2 ([5]) *An embedded surface-link is ribbon if and only if it is equivalent to the closed realizing surface of a band surgery sequence*

$$O \to L \to O,$$

where O is a trivial link and the band surgery $L \to O$ is the inverse of $O \to L$.

4.2 Normal forms for immersed surface-links

Let L be a link and L' a link obtained from L by applying some crossing changes. There is a homotopy $(g_s : M^1 \to \mathbb{R}^3 \mid s \in [0,1])$ of the source circles M^1 of the link into $\mathbb{R}^3$ with $g_0(M^1) = L$ and $g_1(M^1) = L'$ such that each g_s, except $s = 1/2$, is an embedding of M^1 and at $s = 1/2$ intersections occur. We call such a homotopy a *crossing change deformation.* A crossing change deformation from L to L' is denoted by $L \to L'$.

For a crossing change deformation $L \to L'$ by a homotopy $(g_s \mid s \in [0,1])$, the *realizing surface* is a compact oriented surface, say F, properly immersed in $\mathbb{R}^3 \times [a,b]$ defined by:

$$F \cap \mathbb{R}^3 \times \{t\} = g_s(L) \times \{t\} \text{ for } t \in [a,b],$$

where $s = (t-a)/(b-a)$. This realizing surface is denoted by $F(L \to L')_{[a,b]}$.

A link is called an *H-trivial link* with k Hopf links if it is a split union of a trivial link and k Hopf links for some $k \geq 0$. An H-trivial link with k Hopf links can be transformed into a trivial link by k crossing changes. We call a crossing change deformation determined by the crossing changes a *Hopf-splitting deformation.*

Let $\mathcal{L} : L_1 \to L_2 \to \ldots \to L_m$ be a band surgery sequence with H-trivial links L_1 and L_m. The realizing surface $F(\mathcal{L})_{[a+\varepsilon, b-\varepsilon']}$ in $\mathbb{R}^3 \times [a+\varepsilon, b-\varepsilon']$, for some small $\varepsilon, \varepsilon' > 0$, is extended to an oriented surface

$$F(\mathcal{L})^{\times}_{[a,b]} := F(L'_1 \to L_1)_{[a,a+\varepsilon]} \cup F(\mathcal{L})_{[a+\varepsilon,b-\varepsilon']} \cup F(L_m \to L'_m)_{[b-\varepsilon',b]}$$

in $\mathbb{R}^3 \times [a,b]$, where $L'_1 \to L_1$ is the inverse operation of a Hopf-splitting deformation and $L_m \to L'_m$ is a Hopf-splitting deformation. Since links L'_1 and L'_m are trivial links, there exist disk systems $\mathcal{D}$ and $\mathcal{D}'$ in $\mathbb{R}^3$ with $\partial\mathcal{D} = L'_1$ and $\partial\mathcal{D}' = L'_m$. Then we obtain a closed oriented surface

$$\overline{F}(\mathcal{L})^{\times}_{[a,b]} := \mathcal{D} \times \{a\} \cup F(\mathcal{L})^{\times}_{[a,b]} \cup \mathcal{D}' \times \{b\}$$

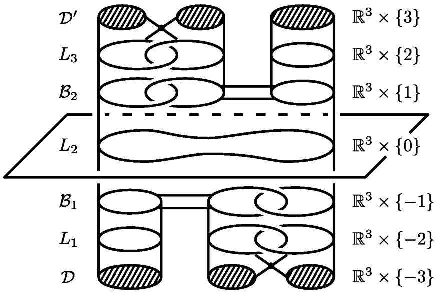

Figure 9: The closed realizing surface $\overline{F}(L_1 \overset{\mathcal{B}_1}{\rightarrow} L_2 \overset{\mathcal{B}_2}{\rightarrow} L_3)^{\times}_{[-3,3]}$

in $\mathbb{R}^3 \times [a, b]$, which we call the *closed realizing surface* of $\mathcal{L}$. See Figure 9. We say that an immersed surface-link is in a *normal form* if it is a closed realizing surface $\overline{F}(\mathcal{L})^{\times}_{[a,b]}$ of a band surgery sequence $\mathcal{L}$.

Theorem 4.3 ([4]) *Every immersed surface-link with μ components and g total genus is equivalent to the closed realizing surface of a band surgery sequence*

$$O \to L_- \to L_0 \to L_+ \to O',$$

where O and O' are H-trivial links, L_- and L_+ are μ-component links and L_0 is a $(\mu+g)$-component link.

A ribbon-clasp surface-link is characterized in terms of normal forms as follows.

Theorem 4.4 ([4]) *An immersed surface-link is ribbon-clasp if and only if it is equivalent to the closed realizing surface of a band surgery sequence*

$$O \to L \to O,$$

where O is an H-trivial link and the band surgery $L \to O$ is the inverse of $O \to L$.

These theorems are generalization of Theorem 4.1 and Theorem 4.2.

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